C.2.1 Optimum Bit Allocation

With b_k denoting the number of bits per sample of the kth subband signal, the quantity

$$b \triangleq \frac{1}{M}\sum_{k=0}^{M-1} b_k \qquad (C.2.2)$$

represents the ***average bit rate,*** that is, average number of bits per subband. Essentially b represents the average number of bits per sample of $x(n)$, transmitted by the decimated analysis bank.

Suppose the average bit rate b is fixed. How should we allocate the bits b_k to the individual subbands so that the total output noise variance (C.2.1) is minimized? Qualitatively speaking, we have to allocate fewer bits ($b_k < b$) for subbands with low energy, and more bits ($b_k > b$) for dominant subbands. An extreme case occurs when most of the energy is in the lowpass subband, in which case we set $b_0 = Mb$ and $b_k = 0$ otherwise. It is clear that this is much more efficient than assigning b bits per sample of the original signal $x(n)$. To address the bit allocation problem more quantitatively, we use the following standard result.

The AM-GM inequality. [Beckenbach and Bellman, 1961]. Given a set of M nonnegative numbers a_k, define the arithmetic mean (AM) and geometric mean (GM) to be

$$\begin{aligned}&\text{Arithmetic mean:} \quad \frac{1}{M}\sum_{k=0}^{M-1} a_k,\\ &\text{Geometric mean:} \quad \Big(\prod_{k=0}^{M-1} a_k\Big)^{1/M}\end{aligned} \qquad (C.2.3)$$

We then have $AM \geq GM$ with equality if and only if $a_0 = a_1 = \ldots = a_{M-1}$. See Problem C.2 for a proof.

Assuming that the filter bank input $x(n)$ is zero mean WSS with variance σ_x^2, the signal $x_k(n)$ [output of $H_k(z)$] is WSS with zero mean and variance

$$\sigma_{x_k}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_k(e^{j\omega})|^2 S_{xx}(e^{j\omega})\, d\omega, \qquad (C.2.4)$$

where $S_{xx}(e^{j\omega})$ is the power spectrum of $x(n)$. The decimated signal $v_k(n)$ is WSS with the same variance $\sigma_{x_k}^2$. With $|H_k(e^{j\omega})|$ as in Fig. C.2-2, one can verify that $\sigma_x^2 = \sum_k \sigma_{x_k}^2/M$ where σ_x^2 is the variance of the input signal $x(n)$. The quantizer variance $\sigma_{q_k}^2$ is related to the variance of $v_k(n)$ in a way similar to (C.1.4), so that

$$\sigma_{q_k}^2 = c \times 2^{-2b_k}\sigma_{x_k}^2. \qquad (C.2.5)$$

Here we have assumed that c is the same for all quantizers. This is valid if the quantizer inputs have identical statistical distribution (e.g., all of them Gaussian).

According to

So the smalle
This quantity
This lower bo
becomes an e

So the optima
sources are eq

so that b_k is
noise variance

The subscript

Align th
step sizes are
us that we mu
representing t

More Explic

The optin
for subband si
has the optim

$b_k = b$

According to (C.2.1) the output noise variance σ_q^2 is the AM of $\sigma_{q_k}^2$ and so we have

$$\begin{aligned} \sigma_q^2 &\geq \Big(\prod_{k=0}^{M-1} \sigma_{q_k}^2\Big)^{1/M} \qquad \text{(AM-GM inequality)} \\ &= c\Big(\prod_{k=0}^{M-1} 2^{-2b_k}\sigma_{x_k}^2\Big)^{1/M} \qquad \text{[using (C.2.5)]} \\ &= c \times \Big(2^{-2\sum b_k/M}\Big)\Big(\prod_{k=0}^{M-1} \sigma_{x_k}^2\Big)^{1/M} \\ &= c \times 2^{-2b}\Big(\prod_{k=0}^{M-1} \sigma_{x_k}^2\Big)^{1/M} \qquad \text{[using (C.2.2)].} \end{aligned} \tag{C.2.6}$$

So the smallest possible value of σ_q^2 is given by the last line of the above equation. This quantity depends on the input signal $x(n)$, and the average number of bits b. This lower bound is achieved if and only if the AM-GM inequality on the first line becomes an equality, which happens if and only if $\sigma_{q_k}^2$ is the same for all k, that is,

$$\sigma_{q_k}^2 = \sigma_q^2 \qquad 0 \leq k \leq M-1. \tag{C.2.7}$$

So the optimal bit allocation strategy is such that *variances of all quantizer noise sources are equalized.* We then have

$$2^{2b_k} = \frac{c \times \sigma_{x_k}^2}{\sigma_q^2}, \tag{C.2.8a}$$

so that b_k is propotional to $\log \sigma_{x_k}^2$. Under this condition, the minimized output noise variance is given by

$$\sigma_{q,SBC}^2 = c \times 2^{-2b}\Big(\prod_{k=0}^{M-1} \sigma_{x_k}^2\Big)^{1/M}. \tag{C.2.8b}$$

The subscript SBC is used because we will soon compare this with other schemes.

Align the least-significant bits! If $\sigma_{q_k}^2$ is the same for all quantizers, the step sizes are identical. The optimal bit allocation strategy (C.2.7) therefore tells us that we must align the rightmost bits (least-significant bits) of the binary words representing the subbands. This is demonstrated in Fig. C.2-3.

More Explicit Expression for Bit Allocation

The optimal bit allocation scheme (C.2.8a) indicates that *more* bits are required for subband signals having *higher* variance. The reconstruction error variance σ_q^2 has the optimum value given by (C.2.8b). Using this we can rearrange (C.2.8a) as

$$b_k = b + 0.5\log_2 \frac{\sigma_{x_k}^2}{\Big(\prod_{i=0}^{M-1} \sigma_{x_i}^2\Big)^{1/M}} \quad \text{(optimal bit allocation).} \tag{C.2.9}$$