1. Histogram Specification

a. Given that $x_i = y_i = 0,1,2,3,4,5,6,7$ and the input probability density function:

$$p_r(x_i) = \begin{cases} \dfrac{x_i}{12} & i = 0,1,2,3 \\ \dfrac{|7 - x_i|}{12} & i = 4,5,6,7 \end{cases}$$

The output probability density function is:

$$p_s(y_i) = \frac{y_i}{28}$$

Find the transformation for mapping the pixels by filling in the table below.

Pixel value	$p_r(x_i)$	Cumulative Distribution of input	$p_s(y_i)$	Cumulative Distribution of output	New Mapping rounding	New Mapping truncation
0	0	0	0	0	0	0
1	0.0833	0.0833	0.0357	0.0357	0	0
2	0.1667	0.2500	0.0714	0.1071	1	1
3	0.2500	0.5000	0.1071	0.2143	2	1
4	0.2500	0.7500	0.1429	0.3571	2	2
5	0.1667	0.9167	0.1786	0.5357	3	3
6	0.0833	1.0000	0.2143	0.7500	4	4
7	0	1.0000	0.2500	1	7	7

b. Suppose that a digital image is histogram equalized. What will happen if the image is equalized again? Does the image stay the same or change and why?

The image stays the same after the second histogram equalization. Let *n* be the total number of pixels in the image and let n_{r_k} be the number with intensity level *r_k*. The histogram equalization transformation is:

$$s_k = T(r_k) = \sum_{j=0}^{k} \frac{n_{r_j}}{n} = \frac{1}{n} \sum_{j=0}^{k} n_{r_j}$$

Since each pixel with value r_k is mapped to value s_k, it follows that $n_{s_j} = n_{r_j}$ A second pass of the histogram equalization algorithm would produce values, v_k according to:

$$v_k = T(s_k) = \sum_{j=0}^{k} \frac{n_{s_j}}{n} = \frac{1}{n} \sum_{j=0}^{k} n_{s_j}$$

but $n_{s_j} = n_{r_j}$, so

$$v_k = T(s_k) = \frac{1}{n} \sum_{j=0}^{k} n_{s_j} = \frac{1}{n} \sum_{j=0}^{k} n_{r_j} = s_k .$$

This shows that a second pass of histogram equalization would yield the same result as the first pass. This assumes negligible round-off error.

2. 2-D Transforms

The basic approach used to compute a 5x5 LoG operator involves a mask of the form:

$$h = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ -1 & -2 & 16 & -2 & -1 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

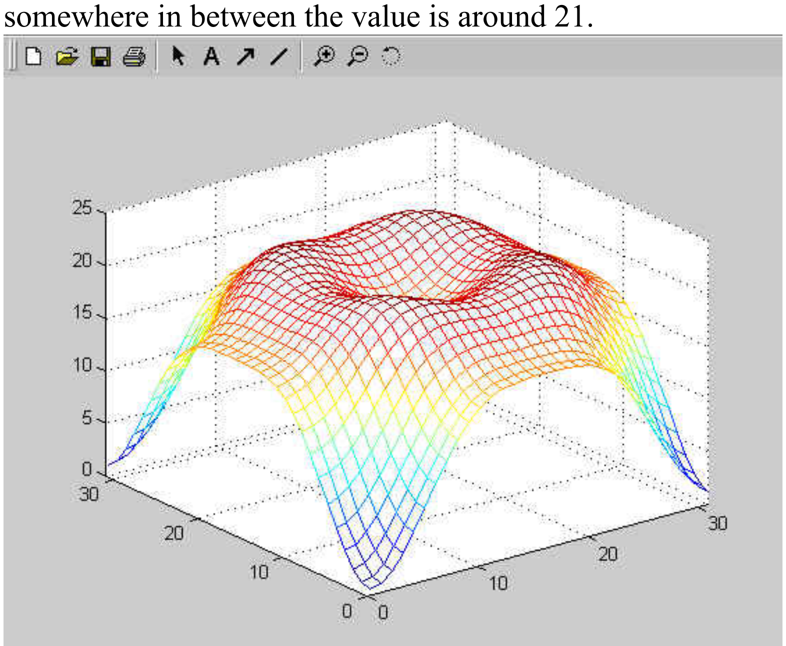
a. Find the Fourier transform, *H(u,v)* of this mask in the frequency domain. (17 points)

b. What type of filter is this, LPF, BPF, or HPF? And why? (8 points)

Assuming that the center of the mask is the (0,0) element:

$$\begin{aligned} H(u,v) &= \sum_{i=-2}^{2}\sum_{k=-2}^{2} h(i,k)\; W_5^{iu+kv} \\ &= 16 - 2\left(e^{-j2\pi u/5} + e^{+j2\pi u/5} + e^{-j2\pi v/5} + e^{+j2\pi v/5}\right) - 1\left(e^{-j2\pi(u+v)/5} + e^{j2\pi(u+v)/5} + e^{-j2\pi(u-v)/5} + e^{j2\pi(u-v)/5}\right) \\ &\qquad -1\left(e^{-j2\pi(2v)/5} + e^{j2\pi(2v)/5} + e^{-j2\pi(2u)/5} + e^{j2\pi(2u)/5}\right) \\ &= 16 - 4(\cos(2\pi u/5) + \cos(2\pi\, v/5)) - 2(\cos(2\pi(u+v)/5) + \cos(2\pi\,(u-v)/5)) \\ &\qquad -2(\cos(4\pi u/5) + \cos(4\pi v/5)) \end{aligned}$$

This is really a BPF as can be shown in the figure below. The low frequencies are in the corners and the high frequencies are in the center. Clearly from the equation, at low frequencies (u=v=0) this is equal to zero. At high frequencies, u= v=2.5 the value is 16, somewhere in between the value is around 21.



3. Geometric Operations

Suppose that you have two digitized images of a canyon wall taken 100 years apart and you wish to detect changes due to erosion by image subtraction. You find a rock that is located at (103, 84) in the first image and at (107,94) in the second. There is also a stump that is located at (433, 504) in the first image and (377,439) in the second. Assume that there has been no distortion beyond translation, rotation, and change in scale.

a. Has there been any translation, rotation, or change in scale between these images? Justify your answers using the data in the problem.(9 points)
b. Write the geometric transformations required to register the second image with the first. (16 points)

Solution:

Yes, there has been translation since the objects have moved in the images. There has also been scaling since the objects are closer each other than they used to be. There is no rotation since the angle of the vector to the stump has not changed.

The first thing to do is find the vector starting at the rock and going to the stump in both images:

Image 1: (433-103,504-84)=(330,420), length_1=534.1

Image 2: (377-107,439-94)=(270,345), length_2=438.1

This gives the scaling between the images:

$$\mathbf{S} = \begin{bmatrix} 270/330 & 0 \\ 0 & 345/420 \end{bmatrix} = \begin{bmatrix} .8182 & 0 \\ 0 & .8214 \end{bmatrix}$$

The next step is to scale image 1 by multiplying each point by **S**, giving the new coordinates:

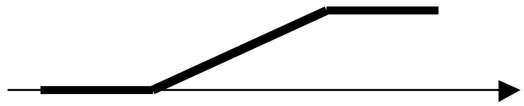
stump1_scaled= 354.2727 414.0000

rock1_scaled = 84.2727 69.0000

The next step is to find the translation by subtracting the positions in image 2 by the scaled position in image 1, this gives translations of: 22.7273 25.0000 in the x and y directions respectively.

4. Edge Detection

a. If a ramp edge (shown below) is used as an edge model, sketch the first and second derivatives. 8 points

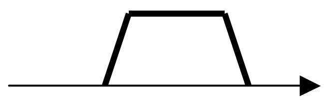


b. Explain how the two types of edge detection methods based on first derivatives and second derivatives work for this type of edge. Describe how these methods work in terms of the edge detector optimality criteria: edge detection, edge localization, and one-response to an edge. 10 points

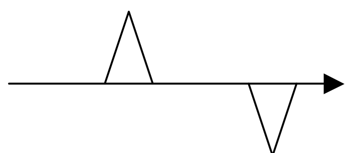
c. Design an edge detection technique that detects a ramp edge as a single edge. 7 points

Solution:

a. First Derivative:



Second Derivative:



b. First derivative edge detectors have a maximum at the edge location. Second derivative edge detectors have a zero crossing at the edge location. The first derivative edge detector will accurately detect the presence of a ramp edge since it will have a sustained response. However it will have problems with edge localization because of the constant response. This method may also give multiple responses to a ramp edge if noise is present and multiple peaks occur in the derivative. Second derivative detectors will have difficulty detecting the presence of an edge due to the length of the zero crossing area. Edge localization should be better since the two responses can be averaged. Second derivatives will give multiple responses since the location of the zero-crossing is not clear cut.

Ideally the response should be in the middle of a ramp edge. So one possible method would use both the first and second derivative.

Step 1: Estimate the first derivative of image. And threshold this value to find adjacent pixels with similar magnitude responses.

Step 2: Estimate the second derivative. Threshold this image to find strong responses.

Step 3: Keep edges that have both first and second derivative support. Fix the edge location between the second derivative responses.